On the Currents induced in a Conductor by the Passage of a Mass of Magnetic Material over it.

By E. S. Bieler, M.Sc., 1851 Scholar of McGill University.

(Communicated by Sir E. Rutherford, F.R.S.—Received April 22, 1921.)

[Plate 1.]

The investigation of the currents induced in a conductor by the passage of a mass of magnetic material over it is a matter of some considerable interest. In the simple case where the mass is in the form of a spherical shell, the problem can be investigated mathematically, and the motion of the spot of a galvanometer connected to such a conductor can be deduced. It has occurred to the writer to compare the results of a mathematical investigation with those of experiments carried out in the laboratory on a small scale. It is the object of the present paper to give a summary of both the theory and the experimental confirmation.

THEORETICAL INVESTIGATION.

A. *The Induced Electromotive Force.*

Consider an iron shell (fig. 1) of internal and external diameters a and b, and of permeability μ, passing with velocity v over an infinitely long straight conductor at depth d below the level of its centre. Take axes through the centre, the Y-axis being vertical and the X- and Z-axes being horizontal, the one perpendicular, the other parallel to the direction of the conductor. Let l, m, n be the direction cosines of F, the earth's field, and let α be the angle between the Z-axis and the direction of motion of the shell.

By the application of spherical harmonics it is easily proved that the magnetic potential at a point (x, y, z) at a distance r from the origin is given by

$$\mathrm{V} = \mathrm{F}(lx+my+nz)(1+\lambda/r^3),$$

where

$$\lambda = -(b^3-a^3)\frac{(2\mu+1)(\mu-1)}{9\mu+2(\mu-1)^2(1-a^3/b^3)}.$$

The term $\mathrm{F}(lx+my+nz)$ is due to the earth's undisturbed field. The other term gives the potential of the disturbance.

Instead of considering the conductor as stationary and the shell as moving past it, it is simpler to reduce the shell to rest, and to consider the conductor as cutting the field due to the disturbance, that is, the field derived from the potential

$$\mathrm{V}' = \mathrm{F}\frac{\lambda(lx+my+nz)}{r^3}.$$

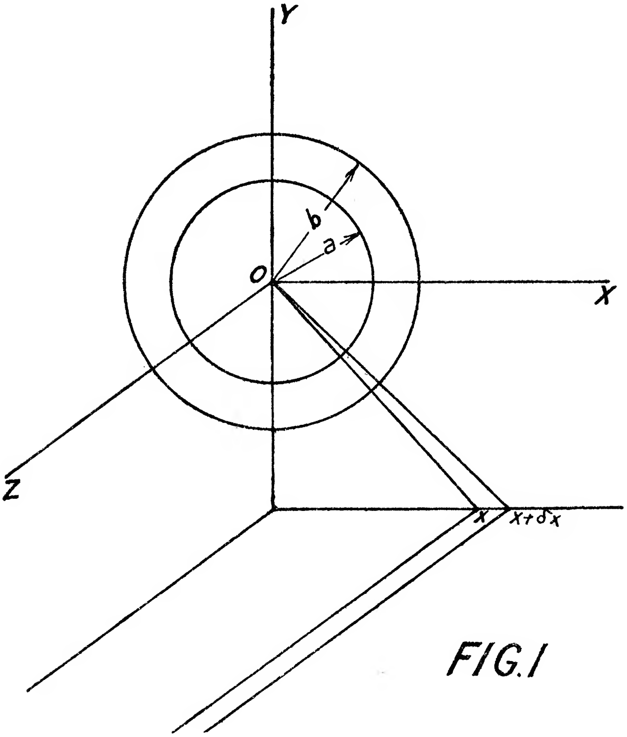


FIG. I

At time t let x be the horizontal distance of the shell from the conductor. In the interval δt, x has increased to $x+\delta x$, and the number N of tubes of induction cut by the conductor in the interval is given by

$$\delta \mathrm{N} = \left(\int_{-\infty}^{\infty} \frac{\partial \mathrm{V}'}{\partial y}\, dz\right) dx.$$

The E.M.F. induced in the conductor is therefore given by

$$\begin{aligned}
\mathrm{E} &= -d\mathrm{N}/dt \\
&= -d\mathrm{N}/dx \,.\, dx/dt \\
&= -\left(\int_{-\infty}^{\infty} \frac{\partial \mathrm{V}'}{\partial y}\, dz\right) v \sin \alpha \\
&= -\mathrm{F}\lambda v \sin \alpha \int_{-\infty}^{\infty} \frac{1}{r^3}\left(m - \frac{3y\,(lx+my+nz)}{r^2}\right) dz \\
&= -2\,\mathrm{F}\lambda v \sin \alpha \,\frac{m\,(x^2-d^2)-2\,lxd}{(x^2+d^2)^2}. \qquad (1)
\end{aligned}$$

B. *The Deflection produced in a critically damped Galvanometer.*

It will be assumed that the galvanometer is critically damped, a condition which can always be secured in a moving coil instrument by introducing a suitable shunt or series resistance.

Let A be the total area of the turns of a galvanometer coil suspended in a field of strength H. Let R be the total resistance in the galvanometer circuit, K the moment of inertia of the suspended system, and τ the directing couple of the suspension for unit angular deflection.

If θ represent the angular deflection of the moving system, i the current at any time, and E the impressed electromotive force, the two following equations are obtained at once from the mechanical and electrical conditions respectively,

$$\mathrm{K}\, d^2\theta/dt^2 - \mathrm{AH}i + \tau\theta = 0, \tag{2}$$

$$\mathrm{E} = \mathrm{R}i + \mathrm{AH}\, d\theta/dt. \tag{3}$$

Eliminating i between (2) and (3), we obtain

$$\frac{d^2\theta}{dt^2} + \frac{\mathrm{A^2H^2}}{\mathrm{RK}}\frac{d\theta}{dt} + \frac{\tau}{\mathrm{K}}\theta = \frac{\mathrm{AH}}{\mathrm{RK}}\mathrm{E}. \tag{4}$$

The galvanometer will be critically damped, provided that R has been so adjusted that

$$(\mathrm{A^2H^2/RK})^2 = 4\tau/\mathrm{K},$$

that is, if

$$\mathrm{R} = \mathrm{A^2H^2}/2\sqrt{(\tau\mathrm{K})}.$$

If we now write $n = \mathrm{A^2H^2}/2\mathrm{RK} = \sqrt{(\tau/\mathrm{K})}$, equation (4) reduces to

$$d^2\theta/dt^2 + 2n\, d\theta/dt + n^2\theta = (2n/\mathrm{AH})\,\mathrm{E},$$

or

$$(\mathrm{D}+n)^2\theta = (2n/\mathrm{AH})\,\mathrm{E}. \tag{5}$$

The values of the coefficients n and $2n/\mathrm{AH}$ are easily obtained by two simple observations on the galvanometer.

For a steady deflection, (5) reduces to $n^2\theta = (2n/\mathrm{AH})\,\mathrm{E}$, that is,

$$\theta/\mathrm{E} = 2/n\mathrm{AH}. \tag{6}$$

Thus, $2/n\mathrm{AH}$ is the sensitivity in radians per unit E.M.F. for a steady deflection.

Also, if the galvanometer is allowed to swing on open circuit, the motion is given by

$$d^2\theta/dt^2 + n^2\theta = 0,$$

and consequently the period T by

$$\mathrm{T} = 2\pi/n. \tag{7}$$

Thus, by means of (6) and (7), we can evaluate the constants of (5) for the voltage sensitivity and the period of the galvanometer.

If we can solve (5) in the case where E is given by equation (1), we are able to deduce the actual motion of the galvanometer spot due to the electromotive force induced in a conductor by the passage of a spherical shell of iron over it.

Substituting in (5) for the value of E given by (1), we obtain

$$(\mathrm{D}+n)^2\theta = -(4n/\mathrm{AH})\,\mathrm{F}\lambda v \sin\alpha\,(m(x^2-d^2)-2lnd)/(x^2+d^2)^2,$$

which, on replacing x by its value $v\sin\alpha\, t = v't$, reduces to

$$\begin{aligned}(\mathrm{D}+n)^2\theta &= -(4n/\mathrm{AH})\,\mathrm{F}\lambda v'\,(m(v'^2t^2-d^2)-2lv'td)/(v'^2t^2+d^2)^2\\ &= -\mathrm{M}\,(m(t^2-p^2)-2ltp)/(t^2+p^2)^2,\end{aligned} \tag{8}$$

where $p = d/v'$ and $\mathrm{M} = 4n\mathrm{F}\lambda/\mathrm{AH}v'$.

Now, it is easily found by the method of "Variation of Parameters,"* that the solution of

$$(\mathrm{D}+n)^2\theta = f(t)$$

is

$$\theta = \{t\int e^{nt}f(t)\,dt - \int te^{nt}f(t)\cdot dt\}\,e^{-nt}. \tag{9}$$

Since $f(t)$ is of the form

$$f(t) = -\mathrm{M}\,(m(t^2-p^2)-2ltp)/(t^2+p^2)^2,$$

the solution will contain the four unknown integrals

$$\int\frac{t^3}{(t^2+p^2)^2}e^{nt}\,dt,\quad \int\frac{t^2}{(t^2+p^2)^2}e^{nt}\,dt,\quad \int\frac{t}{(t^2+p^2)^2}e^{nt}\,dt,\quad \int\frac{1}{(t^2+p^2)^2}e^{nt}\,dt.$$

It would be no simple task to evaluate these for various values of n and p over a considerable range of t, and we shall have to restrict ourselves to the special case where $p = 1/n$.

If we write $nt = y$, the four above integrals reduce to

$$\begin{aligned}\mathrm{I}_1 &= \int_{-\infty}^{y}\frac{y^3}{(y^2+1)^2}e^y\,dy,\\ \mathrm{I}_2 &= \int_{-\infty}^{y}\frac{y^2}{(y^2+1)^2}e^y\,dy,\\ \mathrm{I}_3 &= \int_{-\infty}^{y}\frac{y}{(y^2+1)^2}e^y\,dy,\\ \mathrm{I}_4 &= \int_{-\infty}^{y}\frac{1}{(y^2+1)^2}e^y\,dy,\end{aligned} \tag{10}$$

where the limits have been put in to satisfy the conditions $\theta = 0$, $\dot{\theta} = 0$ at infinity.

The evaluation of these integrals will be made somewhat simpler if we define two other integrals,

$$\mathrm{I}_5 = \int_{-\infty}^{y}\frac{y}{y^2+1}e^y\,dy = \mathrm{I}_1+\mathrm{I}_3,$$

$$\mathrm{I}_6 = \int_{-\infty}^{y}\frac{1}{y^2+1}e^y\,dy = \mathrm{I}_2+\mathrm{I}_4,$$

---

* Forsyth, 'Differential Equations,' 4th ed., § 65.

The first three integrals may be written down in terms of the simpler integrals I_4, I_5, and I_6,

$$I_3 = \int_{-\infty}^{y} \frac{y}{(y^2+1)^2} e^y dy = -\tfrac{1}{2}\int_{-\infty}^{y} e^y d\left(\frac{1}{y^2+1}\right) = \tfrac{1}{2} I_6 - \frac{e^y}{2(y^2+1)},$$

$$I_2 = I_6 - I_4,$$

$$I_1 = I_5 - I_3 = I_5 - \tfrac{1}{2} I_6 + \frac{e^y}{2(y^2+1)}.$$

Substituting y for nt in (9) and (10) and writing $p = 1/n$, we obtain

$$\theta = \{(y/n^2)\int e^y \phi(y)\, dy - (1/n^2)\int y e^y \phi(y)\, dy\} e^{-y}, \qquad (11)$$

where $$\phi(y) = -n^2 M \frac{m(y^2-1)-2ly}{(y^2+1)^2}.$$

If we use this value of $\phi(y)$ in (11), we finally get for θ in terms of I_4, I_5, and I_6,

$$\begin{aligned}\theta &= M(m\{[2yI_4-(y+1)I_6+I_5]e^{-y}+1/(y^2+1)\} \\ &\qquad\qquad + l\{[(y-2)I_6+2I_4]e^{-y}-y/(y^2+1)\}) \\ &= m\theta_1 + l\theta_2. \end{aligned} \qquad (12)$$

The values of I_4, I_5, and I_6 at intervals of 0·6 from $-4·8$ to 7·2 are given in Table I and were calculated as follows:—The integrals between the limits $-\infty$ and 0 were first worked out numerically from formulæ involving the sine- and cosine-integrals, given by Bierens de Haan.* The integrand was then computed for values of the variable at intervals of 0·15 and the integrals evaluated at intervals of 0·6 by the use of Simpson's rule for five equidistant ordinates, viz.,

$$\int_{-2h}^{+2h} y dx = \frac{2}{45} h(7y_1+32y_2+12y_3+32y_4+7y_5),$$

where h is the interval between successive ordinates, and y_1, y_2, y_3, y_4 and y_5 are the values of the integrand at each ordinate. The computations were carried out with the aid of an excellent Brunsviga calculating machine kindly lent by Mr. Richmond, of the Ballistic Office.

In addition to the integrals I_4, I_5 and I_6, Table I gives the values of θ_1/M

---

* 'Verhand. der Kon. Akad. van Wet.,' Part IV, 1858, Table 130, formulæ 4 and 5; Table 131, formula 9; and 'Nouvelles Tables d'Intégrales Définies,' 1867, Table 91, formulæ 7 and 8; Table 92, formula 6.

(Formula 9, Table 131, of the 'Verhandelingen' is correct as it stands, namely,

$$\int_0^\infty \frac{e^{-px}}{(x^2+q^2)^2} dx = 1/2q^3 [\mathrm{ci}(pq) . \sin(pq) - \mathrm{si}(pq) . \cos(pq) + \tfrac{1}{2}\pi . \cos(pq) - pq\{\mathrm{ci}(pq) . \cos(pq) + \mathrm{si}(pq) . \sin(pq) - \tfrac{1}{2}\pi . \sin(pq)\}].$$

The minus sign before the pq has been changed by mistake to a plus sign in the "Nouvelles Tables," Table 92, formula 6.)

and θ_2/M. These deflections are also plotted in the graphs of figs. 2 and 3. In each case the dotted curve represents the E.M.F., and the full curve the resulting galvanometer deflection. Fig. 2*a* corresponds to the case $l=0$, $m=1$; fig. 2*b* to $l=1$, $m=0$; and fig. 3 to $l=\sin\delta/\sqrt{(2)}$, $m=\cos\delta$, where δ is the dip, here equal to 75°.

Table I.

y.	$I_4 = \int_{-\infty}^{y} \frac{1}{(y^2+1)^2} e^y dy$.	$I_5 = \int_{-\infty}^{y} \frac{y}{y^2+1} e^y dy$.	$I_6 = \int_{-\infty}^{y} \frac{1}{y^2+1} e^y dy$.	$\frac{\theta_1}{M}$.	$\frac{\theta_2}{M}$.
−4·8	0·000003	−0·00139	0·000236	0·0216	—
−4·2	0·000019	−0·00283	0·000560	0·0271	−0·0038
−3·6	0·000068	−0·00582	0·001336	0·0352	−0·0109
−3·0	0·000236	−0·01211	0·00327	0·0404	−0·0189
−2·4	0·000862	−0·02559	0·00835	0·0491	−0·0312
−1·8	0·00364	−0·05485	0·02264	0·0658	−0·0520
−1·2	0·01802	−0·11789	0·06632	0·0878	−0·0930
−0·6	0·10232	−0·23793	0·20888	0·0742	−0·1756
0·0	0·48241	−0·34338	0·62145	−0·0352	−0·2781
0·6	1·1344	−0·11439	1·35078	−0·2335	−0·2338
1·2	1·5890	0·70171	2·1644	−0·3356	−0·0563
1·8	1·8481	1·94917	2·9947	−0·2717	0·0875
2·4	2·0175	3·8653	3·9043	−0·1728	0·1527
3·0	2·1483	6·7977	4·9862	−0·0872	0·1527
3·6	2·2639	11·3541	6·3613	−0·0277	0·1440
4·2	2·3771	18·5216	8·2003	0·0087	0·1166
4·8	2·4970	30·0423	10·7579	0·0274	0·0893
5·4	2·6313	48·8208	14·4265	0·0350	0·0662
6·0	2·7917	79·6739	19·8206	0·0364	0·0483
6·6	2·9897	130·831	27·6295	0·0316	0·0329
7·2	3·2427	216·311	36·6947	0·0278	0·0226

EXPERIMENTAL CONFIRMATION.

The truth of the above results was tested by experiments carried out on a small scale in the Electrical Measurements Laboratory at McGill University by the use of a 16-lb. spherical iron shot and a coil of many turns connected to a reflecting galvanometer, whose deflections were registered photographically.

The coil, on a square frame of 125 cm. side, had 200 turns of No. 28B. and S. copper wire, wound close together, to give a circular section of about 1 cm. diameter. Its resistance, 221·5 ohms, was made up to 224 ohms, the critical damping resistance of the galvanometer.

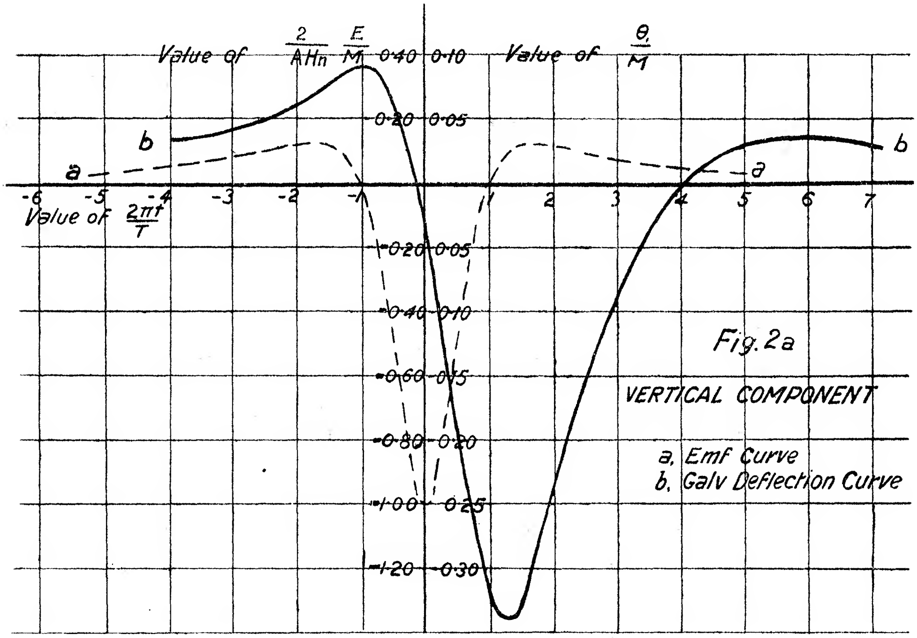
The galvanometer was a moving coil instrument of 8 seconds free period, whose sensitivity, when in series with its critical damping resistance, was 1·63 mm. per microvolt at 1 metre.

The iron sphere, an iron shot kindly lent by Spalding and Co., proved to have excellent magnetic qualities. It exhibited practically no permanent magnetism after lying in the same position in the earth's field for several days.

The galvanometer deflection was recorded by means of a camera consisting of a box with a long horizontal slit. Metal cylinders, actuated by clockwork, unwound a roll of sensitised paper past the slit. A time-mark was made every second by a small pea-lamp connected to a clock. In addition, the pea-lamp circuit was made for an instant as the shot came past three contacts, one directly opposite the side of the coil, and the other two 40 cm. before and after, respectively.

Experiments were carried out for three different positions of the coil.

A. *Vertical Component.*—The shot was made to travel horizontally in a magnetic east and west direction over one side of the coil, pointing magnetic north and south. The horizontal component of the earth's field acted along the side of the coil, and the vertical component alone was effective. In this case, $l = 0$ and $m = \sin \delta = \sin 75^\circ$ in formulæ (1) and (8) (see fig. 2a).



Since the theoretical investigation had only been carried out for the case $p = 1/n$, it was necessary to make sure that the velocity of the shot past the coil was such as to secure this condition. The parameters p and n are connected with the period of the galvanometer, the height of the shot above the coil, and the velocity at right angles to the coil, by the two relations $T = 2\pi/n$ (equation 7) and $p = d/v'$ (equation 8).

Eliminating p and n between these three equations, we have the relation

$$v' = 2\pi d/\mathrm{T}, \tag{13}$$

which gives the required velocity.

Plate 1*a* gives a record of the galvanometer deflection in this case for $d = 14{\cdot}6$ cm. The thick black line is the deflection record, the thin grey lines on either side being only ghosts. The deflection agrees very well with the curve of fig. 2*a*, although the shot velocity was a little low, 10·2 cm. per second instead of 11·6. The maximum deflection measured on the record is 17·0 mm., which is in fair agreement with 14·3 mm., the deflection calculated from formula (12). The difference is probably due to slight permanent magnetisation of the shell.

With the coil in a horizontal position, trouble was experienced at first, due to large disturbances presumably caused by induction from the tramways in the neighbourhood. These disturbances were completely cut down by connecting in series with the coil, and in opposition to it, three turns of wire laid round the laboratory, and enclosing a total area equal to that of all the turns in the coil.

B. *Horizontal Component.*—In this case the coil was placed in a vertical

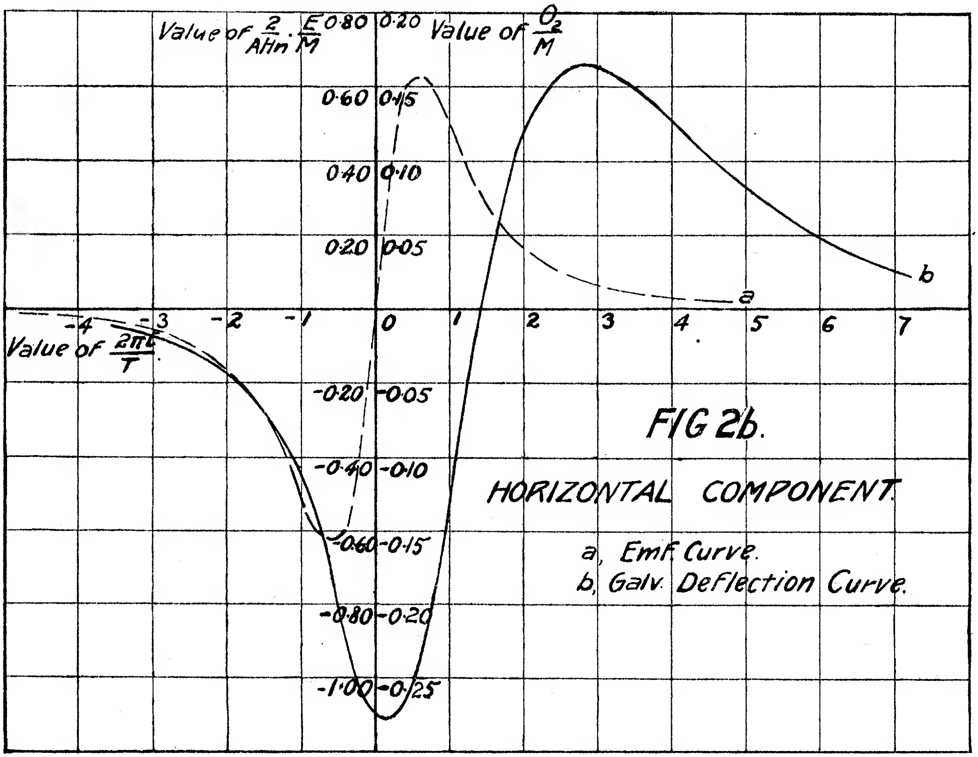


FIG 2b.

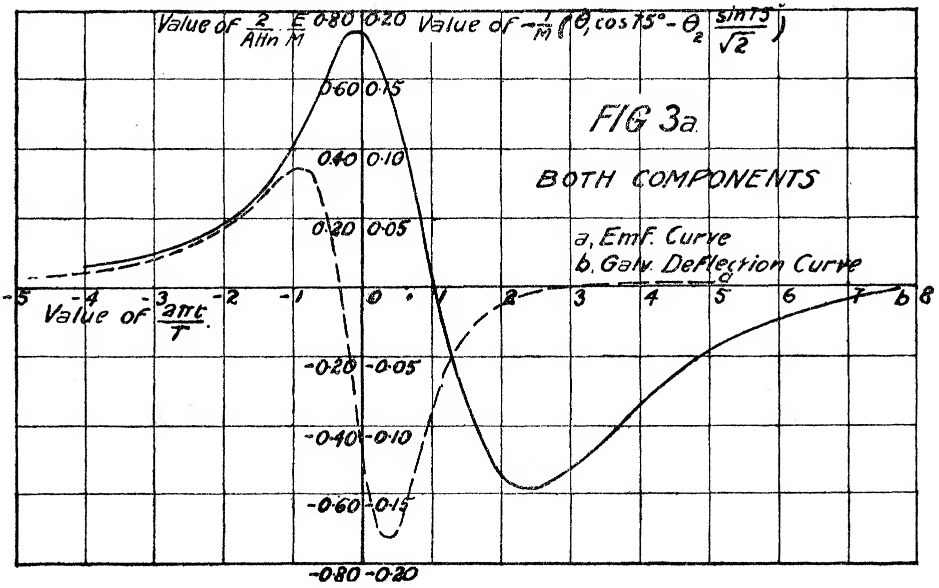
HORIZONTAL COMPONENT

a, Emf Curve.
b, Galv Deflection Curve.

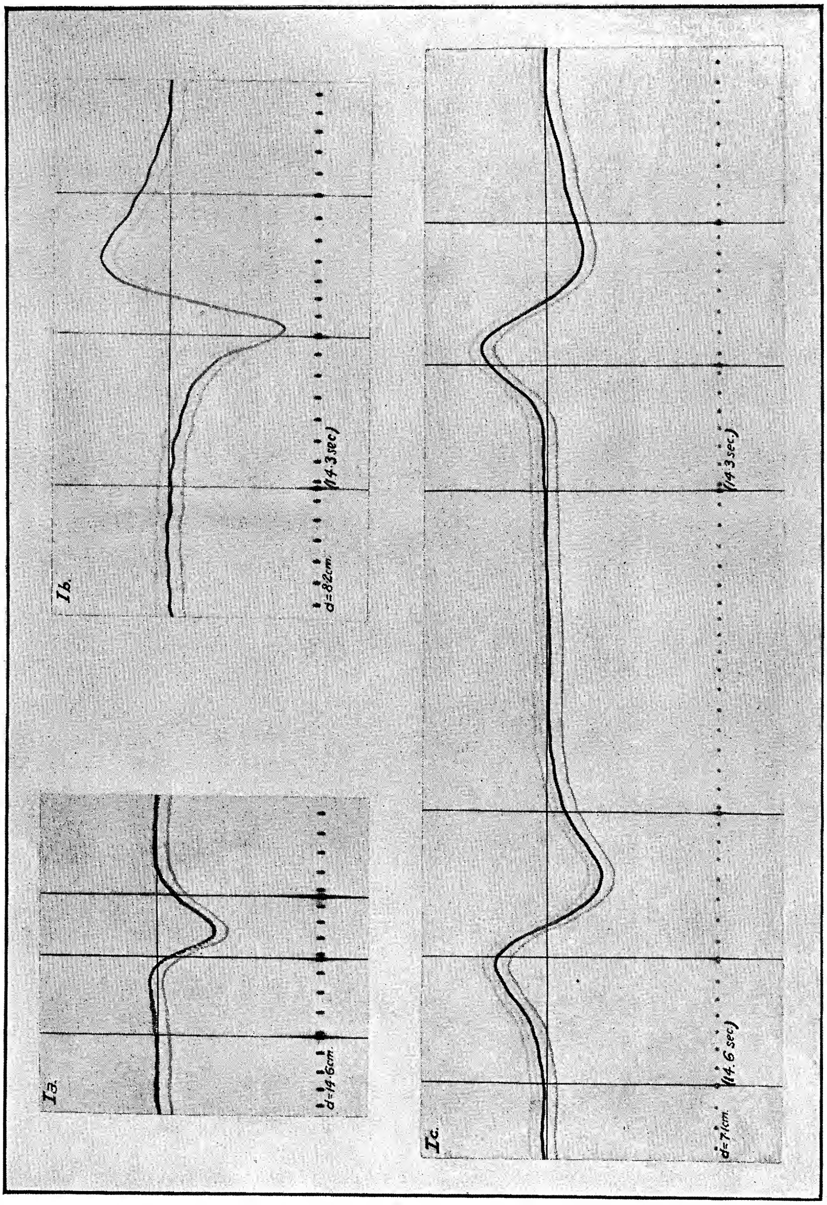
plane pointing magnetic north and south, and the shot was moved up and down past one of its horizontal sides. The horizontal component of the earth's field was directed along the side of the coil, and the vertical component, acting along the direction of motion of the shot, played the part of a virtual horizontal component. This corresponds to $l = \sin\delta = \sin 75^\circ$ and $m = 0$ in formulæ (1) and (8) (see fig. 2*b*).

Here, again, the velocity was determined by means of formula (13). No noticeable tramway disturbances were observed in this position of the coil. Plate 1*b* gives a record of the galvanometer deflection for $d = 8{\cdot}2$ cm. There is good agreement with the calculated deflection shown in fig. 2*b*, although the velocity of the shot was somewhat low. The maximum deflection is 35 mm., against 32·2 mm. calculated from formula (12).

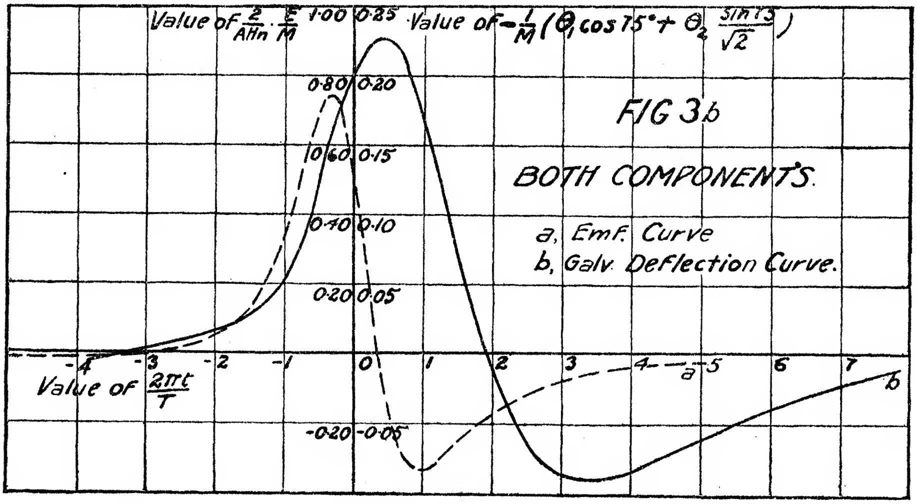
C. *Both Components Combined.*—The coil was again placed in a vertical position, but pointing in a magnetic east and west direction, with its sides inclined at 45° to the horizontal. Here, $l = \sin\delta/\sqrt{(2)} = \sin 75^\circ/\sqrt{(2)}$, $m = \cos\delta$. As may be seen from equation (1), the E.M.F. due to the component parallel to the direction of motion of the shot changes sign with x, whereas that due to the component perpendicular to both the side of the coil and the direction of motion does not. The form of the E.M.F. curve and of the deflection curve will thus depend on the direction of motion of the shot. This is illustrated by the curves of fig. 3 and the record of Plate 1*c*. In fig. 3*a*, and the first deflection in the plate, the positive maximum comes just before the zero-line, and is approximately equal to the negative maximum.



Bieler. *Roy. Soc. Proc.*, *A*, *vol.* 100, *Pl.* 1.



In fig. 3*b*, and the second deflection in the plate, the positive maximum comes just before the zero-line, and is over twice the negative maximum.



Conclusion.

We have now shown that the theory agrees with experimental evidence, and we may draw further conclusions as to the way in which the galvanometer deflection varies with different factors.

The galvanometer deflection is proportional to a factor, M, given by

$$M = \frac{4nF\lambda}{AHv'} = 2 \cdot \frac{2}{AHn} \cdot F \cdot \lambda \cdot \frac{1}{d} \cdot \frac{2\pi}{T}.$$

Thus, if the velocity is that given by $v' = 2\pi d/T$ (equation 13), the deflection will be proportional to—

(*a*) The voltage sensitivity, $2/AHn$, of the galvanometer;

(*b*) The earth's field, F; and

(*c*) The constant λ, which depends on the amount and distribution of magnetic material in the shell,

and will be inversely proportional to—

(*a*) The depth *d* of the conductor below the level of the shell; and

(*b*) The period T of the galvanometer.

Further, it has been found experimentally, and pointed out in the case of Plate 1, *a* and *b*, that a small variation in the velocity produces only a small effect on the deflection. The above results will therefore hold approximately even when the velocity is not that given by equation (13).

The theory also throws light upon the connection that may be expected between the deflection of a critically damped galvanometer, and the E.M.F.

variation which caused it. Two points appear at once on examination of the theoretical curves of figs. 2 and 3:—

(*a*) The galvanometer deflection does not give a perfectly faithful, but a slightly distorted, reproduction of the E.M.F. curve.

(*b*) The galvanometer deflection lags behind the E.M.F. by an amount here equal to about one-fourth of the period.

Both these deviations from a true representation of the E.M.F. variation are due to the effect of the inertia of the moving system of the galvanometer.

On some New Formulæ for the Numerical Calculation of the Mutual Induction of Coaxial Circles.

By Louis V. King, D.Sc., Macdonald Professor of Physics, McGill University, Montreal.

(Communicated by Prof. A. S. Eve, F.R.S. Received April 13, 1921.)

SECTION 1.—*Note on the Direct Numerical Calculation of Elliptic Integrals.*

The importance of realising rapid and accurate methods of calculating the elliptic integrals, now denoted by

$$\mathrm{F}(\phi, k) = \int_0^\phi d\phi/\Delta(\phi, k), \qquad \mathrm{E}(\phi, k) = \int_0^\phi \Delta(\phi, k)\, d\phi, \tag{1}$$

where

$$\Delta(\phi, k) = \sqrt{(1-k^2 \sin^2 \phi)},$$

was first remarked by Euler* (1766), although it was not until several years later that Landen† (1775) discovered in geometrical form the transformation which forms the basis of existing methods of numerical calculation of the elliptic integrals.

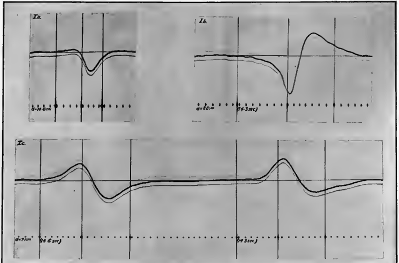
A method of successive transformations for the ultimate reduction of the algebraic forms of the elliptic integrals to elementary forms was published by Lagrange‡ in 1784–5. This memoir contains an exposition of the scales of arithmetico-geometrical means and discusses their use in calculating the

---

* Euler, 'Novi Comm. Acad. Sc. Petrop.,' vol. 10 (1766).

† Landen, 'Phil. Trans. Roy. Soc.,' vol. 65, p. 283 (1775); also, 'Mathematical Memoirs,' London, 1780.

‡ Lagrange, "Sur une nouvelle méthode de calcul intégral pour les differentielles affectées d'un radical carré sous lequel la variable ne passe pas le quatrième degré," 'Mém. de l'Acad. Roy. des Sc. de Turin,' vol. 2 (1784–5); 'Œuvres,' vol. 2, pp. 253–312 (Gauthier-Villars, Paris, 1868).


Ia.
d=14·6km
Ib.
d=82km
(14·3sec)
Ic.
d=7km
(14·6sec)
(14·3sec)